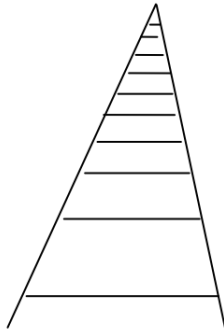


# 4. CROSS RATIO

## §4.1. Euclidean Cross Ratio

Very few quantitative properties of an object remain fixed when it's drawn in perspective. For example the following is a perspective drawing of a railway track.



The picture is very much smaller than the original, so lengths and areas are not preserved. The sleepers on the actual track are at right angles to the lines yet in the perspective drawing they're not, so angles are not preserved.

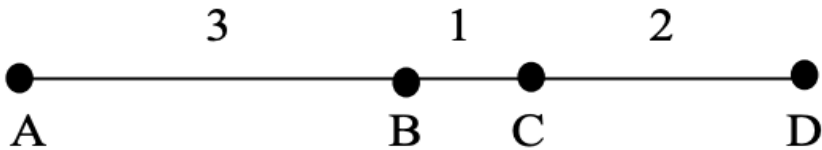
Ratios of lengths along a line are not preserved because the sleepers are equally spaced while they get closer and closer in the perspective drawing. But *ratios of ratios* along any line *are* preserved.

Suppose A, B, C, D are four collinear points. Choose a direction along the line as the positive direction (so that distances measured in the opposite direction are

treated as negative). The cross-ratio of these four points, in the given order, is defined to be:

$\mathfrak{R}(A, B; C, D) = \frac{AC}{BC} \bigg/ \frac{AD}{BD}$  where AC, BC etc denote the signed distances between the respective points. (Clearly the cross ratio is independent of the chosen direction, but does depend on the order in which the points are taken.)

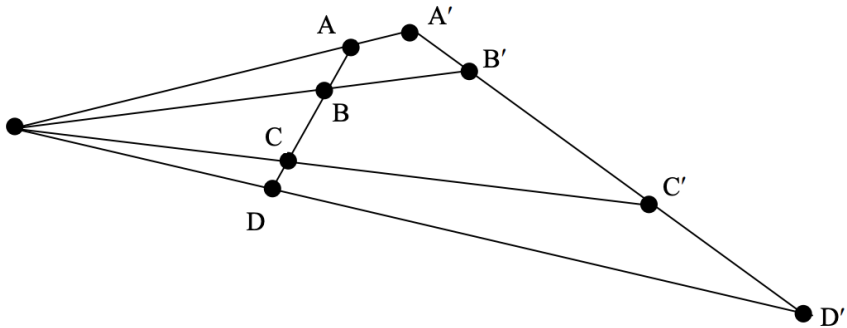
**Example 1:**



$$\mathfrak{R}(A, B; C, D) = \frac{AC}{BC} \frac{AD}{BD} = \frac{4}{1} \frac{6}{3} = 2.$$

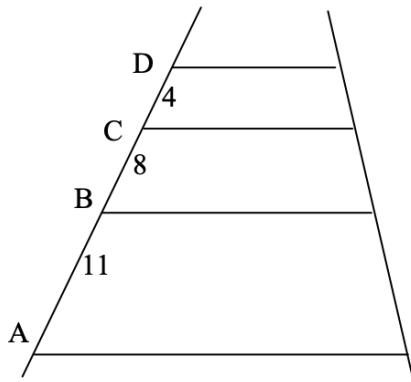
As mentioned earlier, the cross ratio depends on the order of the points. In the above example  $\mathfrak{R}(C, A; D, B) = \frac{CD}{AD} \bigg/ \frac{CB}{AB} = \frac{2}{6} \bigg/ \frac{-1}{3} = -1$ .

The fact (which we have yet to prove) that cross ratios are preserved when things are drawn in perspective means that if four collinear points are projected from any point onto another line, the image points have the same cross ratio as the original ones.



$$\mathfrak{R}(A, B; C, D) = \mathfrak{R}(A', B'; C', D')$$

**Example 2:** Show that the following drawing cannot be a true perspective drawing of any piece of railway track.



**Solution:** The cross ratio  $\mathfrak{R}(A, B; C, D)$  on the drawing is  $\frac{19}{8} / \frac{23}{12} = \frac{57}{46}$ .

The corresponding four points on the original railway track have cross ratio  $\frac{2}{1} / \frac{3}{2} = 4/3$ . Since these are

different this picture can't be an accurate picture of the railway track.

## §4.2. Projective Cross Ratio

**Theorem 1 (COOPER):** The scalar  $\lambda$  in the Collinearity Lemma is unique (ie depends only on the four points and the order in which they are listed.)

**Proof:** Suppose, using the Collinearity Lemma twice,

$$P = \langle \mathbf{p} \rangle = \langle \mathbf{p}' \rangle;$$

$$Q = \langle \mathbf{q} \rangle = \langle \mathbf{q}' \rangle;$$

$$R = \langle \mathbf{p} + \mathbf{q} \rangle = \langle \mathbf{p}' + \mathbf{q}' \rangle;$$

$$S = \langle \lambda \mathbf{p} + \mathbf{q} \rangle = \langle \lambda' \mathbf{p}' + \mathbf{q}' \rangle.$$

Then for some scalars  $\alpha, \beta, \gamma, \delta$ :

$$\mathbf{p}' = \alpha \mathbf{p};$$

$$\mathbf{q}' = \beta \mathbf{q};$$

$$\mathbf{p}' + \mathbf{q}' = \gamma(\mathbf{p} + \mathbf{q});$$

$$\lambda' \mathbf{p}' + \mathbf{q}' = \delta(\lambda \mathbf{p} + \mathbf{q}).$$

Hence  $\alpha \mathbf{p} + \beta \mathbf{q} = \gamma \mathbf{p} + \gamma \mathbf{q}$ , and since  $\mathbf{p}, \mathbf{q}$  are linearly independent,  $\alpha = \beta = \gamma$ .

Also  $\lambda' \alpha \mathbf{p} + \beta \mathbf{q} = \delta(\lambda \mathbf{p} + \mathbf{q})$  and so

$$\lambda' \alpha = \delta \lambda;$$

$$\beta = \delta.$$

Thus  $\delta = \beta = \alpha$  and so  $\lambda' \alpha = \alpha \lambda$ .

Now  $\alpha \neq 0$  (since  $\mathbf{p}' \neq \mathbf{0}$ ) and so  $\lambda' = \lambda$ .

**Definition:** The value of  $\lambda$  in the Collinearity Lemma is the **Projective Cross Ratio** of the four points,  $\mathfrak{R}(P, Q; R, S)$ .

### Example 3:

Find the projective cross ratio  $\mathfrak{R}(P, Q; R, S)$  when

$$P = \langle(1, 2, 3)\rangle, Q = \langle(4, 5, 6)\rangle,$$

$$R = \langle(2, 1, 0)\rangle \text{ and } S = \langle(2, 3, 4)\rangle.$$

**Solution:** Since we're now working in the projective plane we can't use distances. Instead we use the Collinearity Lemma.

$$\text{Write } (2, 1, 0) = x(1, 2, 3) + y(4, 5, 6).$$

Hence:

$$x + 4y = 2;$$

$$2x + 5y = 1;$$

$$3x + 6y = 0.$$

Normally three equations in two variables are inconsistent. The fact that they're consistent here reflects the fact that  $P, Q, R$  are collinear.

$$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_2 - 2R_1, R_3 - 3R_1 \quad R_3 - 2R_2, R_2 \div (-3)$$

giving the solution  $y = 1, x = -2$ .

$$\text{Thus } (2, 1, 0) = (-2, -4, -6) + (4, 5, 6).$$

[In this case we could have simply substituted  $x = -2y$  from the third equation to give  $y = 1$  from the others.]

So take  $\mathbf{p} = (-2, -4, -6)$  and  $\mathbf{q} = (4, 5, 6)$ .

Then  $P = \langle\mathbf{p}\rangle, Q = \langle\mathbf{q}\rangle, R = \langle\mathbf{p} + \mathbf{q}\rangle$ .

Now write  $(2, 3, 4) = x(-2, -4, -6) + y(4, 5, 6)$ .

Proceeding as above we get  $x = -1/3, y = 1/3$ .

Hence  $(2, 3, 4) = -(1/3)\mathbf{p} + (1/3)\mathbf{q}$ .

Thus  $3(2, 3, 4) = -\mathbf{p} + \mathbf{q}$ .

So  $S = \langle -\mathbf{p} + \mathbf{q} \rangle$ , which has the form  $\langle \lambda\mathbf{p} + \mathbf{q} \rangle$

where  $\lambda = -1$ .

Hence the projective cross ratio is  $\mathfrak{R}(P, Q; R, S) = -1$ .

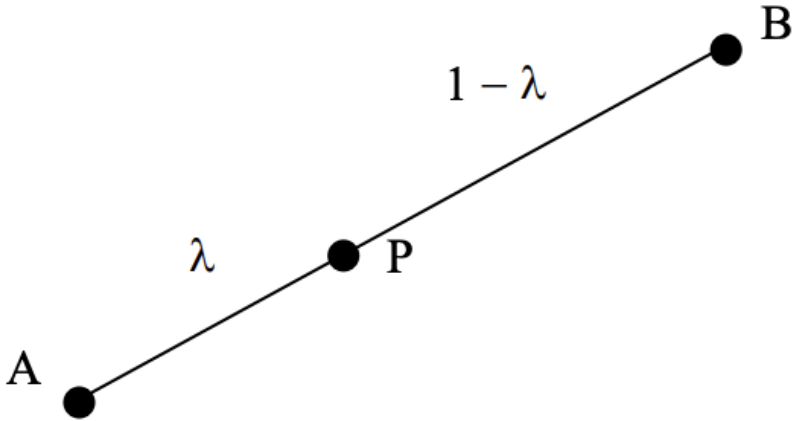
### **§4.3. Reconciliation of Euclidean and Projective Cross Ratios**

We must now show that the concepts of Euclidean and Projective Cross Ratios are equivalent. We do this by showing that if the Euclidean Plane is embedded in the Real Projective Plane and four collinear projective points correspond to four collinear ordinary points, then the Projective Cross Ratio and Euclidean Cross Ratio are equal.

But first we need to recall a fact from vector geometry about points that divide a line segment in a given ratio.

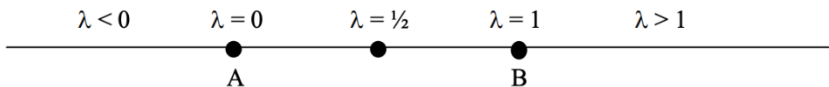
If  $\mathbf{a}, \mathbf{b}$  represent two distinct points  $A, B$  in  $\mathbb{R}^3$  the typical point on the line  $AB$  is represented by:

$\mathbf{p} = (1 - \lambda) \mathbf{a} + \lambda \mathbf{b}$  where  $\lambda = \frac{AP}{AB}$  and where AP and AB represent the directed lengths of the respective sub-intervals (taking the direction from A to B as positive).

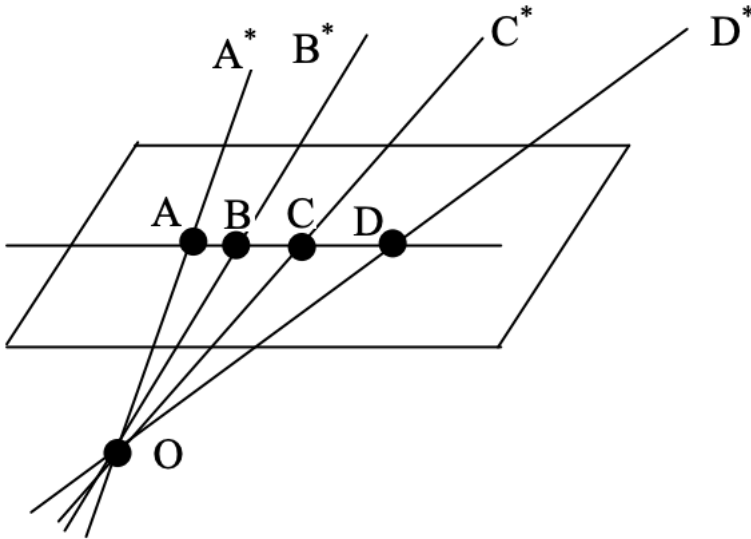


Special cases are:  $\lambda = 0$ , where  $P = A$ ,  $\lambda = 1/2$  (where  $P$  is the midpoint of  $AB$ ) and  $\lambda = 1$  (where  $P = B$ ).

For  $\lambda > 1$   $P$  lies beyond  $B$  (on the opposite side to  $A$ ) and for  $\lambda < 0$ ,  $P$  lies beyond  $A$  (on the opposite side to  $B$ .) As  $\lambda$  ranges from  $-\infty$  to  $+\infty$  the point  $P$  traverses the line in the direction from  $A$  to  $B$ .



**Theorem 2 (COOPER):** Suppose  $\pi$  is a Euclidean Plane that does not pass through the origin. If  $A, B, C, D$  are distinct collinear points on  $\pi$  and  $A^*, B^*, C^*, D^*$  are the corresponding projective points  $OA, OB, OC, OD$  then the projective cross ratio of  $A^*, B^*, C^*, D^*$  is equal to the Euclidean Cross Ratio of  $A, B, C, D$ .



**Proof:** By the Collinearity Lemma we can choose vectors  $\mathbf{a}, \mathbf{b}$  and a scalar  $\lambda$  such that:

$$A^* = \langle \mathbf{a} \rangle;$$

$$B^* = \langle \mathbf{b} \rangle;$$

$$C^* = \langle \mathbf{a} + \mathbf{b} \rangle;$$

$$D^* = \langle \lambda \mathbf{a} + \mathbf{b} \rangle.$$

Moreover  $\lambda$  is the projective cross ratio of the four projective points  $A^*, B^*, C^*, D^*$ .

Now  $A, B, C, D$  lie on the vectors  $\mathbf{a}, \mathbf{b}, \mathbf{a} + \mathbf{b}, \lambda\mathbf{a} + \mathbf{b}$  so they're represented by the vectors

$\alpha\mathbf{a}, \beta\mathbf{b}, \gamma(\mathbf{a} + \mathbf{b}), \delta(\lambda\mathbf{a} + \mathbf{b})$  for certain scalars  $\alpha, \beta, \gamma, \delta$ .

Since  $C, D$  lie on the line  $AB$  we may write

$$\gamma(\mathbf{a} + \mathbf{b}) = (1 - s)\alpha\mathbf{a} + s\beta\mathbf{b}$$

$$\text{and } \delta(\lambda\mathbf{a} + \mathbf{b}) = (1 - t)\alpha\mathbf{a} + t\beta\mathbf{b}$$

$$\text{where } s = \frac{AC}{AB} \text{ and } t = \frac{AD}{AB}.$$

Now  $\mathbf{a}, \mathbf{b}$  are linearly independent so, equating coefficients:

$$\gamma = \alpha(1 - s)$$

$$\gamma = \beta s$$

$$\delta\lambda = \alpha(1 - t)$$

$$\delta = \beta t$$

from which it follows that  $\frac{s}{1 - s} = \frac{\alpha}{\beta}$  and  $\frac{t}{1 - t} = \frac{\alpha}{\beta\lambda}$ .

Since  $s = \frac{AC}{AB}$ ,  $1 - s = \frac{CB}{AB}$ . Similarly  $1 - t = \frac{DB}{AB}$ .



Thus  $\frac{AC}{BC} = -\left(\frac{s}{1 - s}\right)$  and  $\frac{AD}{BD} = -\left(\frac{t}{1 - t}\right)$  and so

$$\frac{AC}{BC} \bigg/ \frac{AD}{BD} = \frac{s}{1 - s} \cdot \frac{1 - t}{t} = \frac{\alpha}{\beta} \cdot \frac{\beta\lambda}{\alpha} = \lambda.$$

**Example 4:** Find the cross ratio  $\mathfrak{R}(P, Q; R, S)$  for the four projective points in Example 3 by using the corresponding points in the plane  $x = 1$ .

**Solution:** The four projective points are:

$$P = \langle(1, 2, 3)\rangle, Q = \langle(4, 5, 6)\rangle, R = \langle(2, 1, 0)\rangle \text{ and } S = \langle(2, 3, 4)\rangle.$$

The corresponding points on the plane  $x = 1$  are:

$$P' = (1, 2, 3), Q' = (1, 5/4, 3/2), R' = (1, 1/2, 0) \text{ and } S' = (1, 3/2, 2).$$

Using just the  $y$ - and  $z$ -coordinates we can write these as:

$$P' = (2, 3), Q' = (5/4, 3/2), R' = (1/2, 0), S' = (3/2, 2).$$

Thus  $\mathfrak{R}(P, Q; R, S) = \mathfrak{R}(P', Q'; R', S')$

$$\begin{aligned} &= \frac{P'R'}{Q'R'} \bigg/ \frac{P'S'}{Q'S'} \\ &= \frac{P'R' \cdot Q'S'}{Q'R' \cdot P'S'} \\ &= - \frac{P'R' \cdot S'Q'}{Q'R' \cdot P'S'} \\ &= - \frac{\sqrt{\left(\frac{3}{2}\right)^2 + 3^2} \cdot \sqrt{\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{2}\right)^2}}{\sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{3}{2}\right)^2} \cdot \sqrt{\left(\frac{1}{2}\right)^2 + 1^2}} \\ &= - \frac{\sqrt{\frac{45}{4}} \cdot \sqrt{\frac{5}{16}}}{\sqrt{\frac{45}{16}} \cdot \sqrt{\frac{5}{4}}} = -1. \end{aligned}$$

## §4.4. Cross Ratio of Rearrangements

The cross ratio of four projective points depends on the order in which they're specified. There are  $4! = 24$  orders in which they could be given. However there are at most 6 different values of the cross ratio that can be obtained because certain rearrangements don't change the value of the cross ratio.

**Theorem 3:** If  $\mathfrak{R}(P, Q; R, S) = \lambda$  then

$$\mathfrak{R}(P, Q; S, R) = 1/\lambda;$$

$$\mathfrak{R}(R, S; P, Q) = \lambda;$$

$$\mathfrak{R}(P, R; Q, S) = 1 - \lambda.$$

**Proof:** Let  $P = \langle \mathbf{p} \rangle$ ,  $Q = \langle \mathbf{q} \rangle$ ,  $R = \langle \mathbf{p} + \mathbf{q} \rangle$ ,  $S = \langle \lambda \mathbf{p} + \mathbf{q} \rangle$ .

Let  $\mathbf{p}_1 = \lambda \mathbf{p}$ ,  $\mathbf{q}_1 = \mathbf{q}$ .

Then  $P = \langle \mathbf{p}_1 \rangle$ ,  $Q = \langle \mathbf{q}_1 \rangle$ ,  $S = \langle \mathbf{p}_1 + \mathbf{q}_1 \rangle$ ,  $R = \langle (1/\lambda)\mathbf{p}_1 + \mathbf{q}_1 \rangle$ .

Hence  $\mathfrak{R}(P, Q; S, R) = 1/\lambda$ .

Let  $\mathbf{p}_1 = \mathbf{p} + \mathbf{q}$ ,  $\mathbf{q}_1 = -\lambda \mathbf{p} - \mathbf{q}$ .

Then  $R = \langle \mathbf{p}_1 \rangle$ ,  $S = \langle \mathbf{q}_1 \rangle$ ,  $P = \langle (1 - \lambda)\mathbf{p} \rangle = \langle \mathbf{p}_1 + \mathbf{q}_1 \rangle$ ,

$Q = \langle (\lambda - 1)\mathbf{q} \rangle = \langle \lambda \mathbf{p}_1 + \mathbf{q}_1 \rangle$ .

Hence  $\mathfrak{R}(R, S; P, Q) = \lambda$ ;

Let  $\mathbf{p}_1 = -\mathbf{p}$ ,  $\mathbf{q}_1 = \mathbf{p} + \mathbf{q}$ .

Then  $P = \langle \mathbf{p}_1 \rangle$ ,  $R = \langle \mathbf{q}_1 \rangle$ ,  $Q = \langle \mathbf{p}_1 + \mathbf{q}_1 \rangle$  and

$$S = \langle (1 - \lambda)\mathbf{p}_1 + \mathbf{q}_1 \rangle.$$

Hence  $\mathfrak{R}(P, R; Q, S) = 1 - \lambda$ .

**Theorem 4:** If  $\mathfrak{R}(P, Q; R, S) = \lambda$  then the 24 arrangements of these points can be partitioned into six classes of four, each class corresponding to one of the values  $\lambda, 1 - \lambda, \frac{1}{\lambda}, \frac{1}{1 - \lambda}, \frac{\lambda - 1}{\lambda}$  and  $\frac{\lambda}{\lambda - 1}$  as cross ratio.

**Proof:**  $\mathfrak{R}(P, Q; S, R) = \frac{1}{\lambda}$ ;  $\mathfrak{R}(P, R; Q, S) = 1 - \lambda$ ;

$$\mathfrak{R}(P, R; S, Q) = \frac{1}{\mathfrak{R}(P, R; Q, S)} = \frac{1}{1 - \lambda};$$

$$\mathfrak{R}(P, S; Q, R) = 1 - \mathfrak{R}(P, Q; S, R) = 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda};$$

$$\mathfrak{R}(P, S; R, Q) = 1 - \mathfrak{R}(P, R; S, Q) = 1 - \frac{1}{1 - \lambda} = \frac{\lambda}{\lambda - 1};$$

$$\begin{aligned} \mathfrak{R}(Q, P; R, S) &= 1 - \mathfrak{R}(Q, R; P, S) = 1 - \mathfrak{R}(P, S; Q, R) \\ &= 1 - \frac{\lambda}{\lambda - 1} = \frac{1}{\lambda}. \end{aligned}$$

The verification for the remaining arrangements is left as an exercise.

**Example 5:** If  $\mathfrak{R}(A, B; C, D) = \frac{1}{4}$  find  $\mathfrak{R}(A, D; B, C)$ .

**Solution:**  $\mathfrak{R}(A, B; D, C) = 4$ .

$$\mathfrak{R}(A, D; B, C) = 1 - 4 = -3.$$

**Example 6:** Suppose  $A = \langle(3, 1, 0)\rangle$ ,  $B = \langle(4, 5, 3)\rangle$ ,  $C = \langle(1, 0, 1)\rangle$ ,  $D = \langle(3, 4, 5)\rangle$  are projective points in the projective plane  $\wp(\mathbb{Z}_7)$ . Calculate  $\mathfrak{R}(A, B; C, D)$  and  $\mathfrak{R}(B, D; A, C)$ .

**Solution:**  $(1, 0, 1) = 3(3, 1, 0) + 5(4, 5, 3)$  (this requires some work).

Let  $\mathbf{u} = 3(3, 1, 0) = (2, 3, 0)$  and  $\mathbf{v} = 5(4, 5, 3) = (6, 4, 1)$ .

Then  $A = \langle \mathbf{u} \rangle$ ,  $B = \langle \mathbf{v} \rangle$ ,  $C = \langle \mathbf{u} + \mathbf{v} \rangle$ .

Finally  $(3, 4, 5) = 4(2, 3, 0) + 5(6, 4, 1)$   
 $= 5(5(2, 3, 0) + (6, 4, 1)) = 5(5\mathbf{u} + \mathbf{v})$ .

So  $D = \langle 5\mathbf{u} + \mathbf{v} \rangle$  and hence  $\mathfrak{R}(A, B; C, D) = 5$ .

Hence  $\mathfrak{R}(B, A; C, D) = 1/5 = 3 \pmod{7}$ .

Thus  $\mathfrak{R}(B, A; D, C) = 1/3 = 5$ .

Thus  $\mathfrak{R}(B, D; A, C) = 1 - 5 = 3 \pmod{7}$ .

## EXERCISES FOR CHAPTER 4

**Exercise 1:** (i) If  $A = (0, 1)$ ,  $B = (3, 7)$ ,  $C = (5, 11)$  and  $D = (6, 13)$  find  $\mathfrak{R}(A, B; C, D)$ .

(ii) For the above four points find  $\mathfrak{R}(C, B; A, D)$ .

**Exercise 2:** In the projective plane  $\mathcal{P}(\mathbb{Z}_7)$  let

$A = \langle (1, 1, 0) \rangle$ ,  $B = \langle (5, 2, 3) \rangle$ ,  $C = \langle (0, 2, 5) \rangle$  and

$D = \langle (4, 5, 6) \rangle$ . Find  $\mathfrak{R}(A, B; C, D)$ .

**Exercise 3:**

Let  $A, B, C$  and  $D$  be distinct collinear points such that

$\mathfrak{R}(A, B; C, D) = -2$ . Use the Collinearity Lemma to find

$\mathfrak{R}(C, A; B, D)$ .

**Exercise 4:**

Suppose A, B, C, D, E are five distinct collinear points.

$$\text{Prove that } \mathfrak{R}(B, C; D, E) = \frac{\mathfrak{R}(A, C; B, E)}{\mathfrak{R}(A, C; B, D)}.$$

[**HINT:** Use the Collinearity Lemma on A, B, C, D and on B, C, D, E.]

## SOLUTIONS FOR CHAPTER 4

$$\begin{aligned} \text{Exercise 1: (i) } \mathfrak{R}(A, B; C, D) &= \frac{AC/BC}{AD/BD} = \frac{\sqrt{125}/\sqrt{20}}{\sqrt{180}/\sqrt{45}} \\ &= \frac{5\sqrt{5}/2\sqrt{5}}{6\sqrt{5}/3\sqrt{5}} = \frac{5/2}{6/3} = \frac{5}{4}. \end{aligned}$$

[**NOTE:** We could have avoided the square roots by projecting the four points onto the  $x$ -axis, but the relevant theory has to wait until the next chapter.]

(ii) Rather than work it out afresh we can proceed as follows by Theorem 3:

$\mathfrak{R}(A, B; C, D) = 5/4$  so  $\mathfrak{R}(C, D; A, B) = 5/4$  and hence

$\mathfrak{R}(C, D; B, A) = 4/5$  and so

$\mathfrak{R}(C, B; D, A) = 1 - 4/5 = 1/5$  and so  $\mathfrak{R}(C, B; A, D) = 5$ .

**Exercise 2:**  $\mathfrak{R}(A, B; C, D) = -2$  so  $\mathfrak{R}(C, D; A, B) = -2$

and hence  $\mathfrak{R}(C, D; B, A) = -1/2$  and so

$\mathfrak{R}(C, B; D, A) = 1 - (-1/2) = 3/2$  and so, finally,

$\mathfrak{R}(C, B; A, D) = 2/3$ .

**Exercise 3:** Let  $(0, 6, 5) = x(1, 1, 0) + y(5, 2, 3)$ .

$$\text{Then } \begin{cases} x + 5y = 0 \\ x + 2y = 2 \\ 3y = 5 \end{cases}.$$

The last equation gives  $3y = 5 = 12$  so  $y = 4$  and  $x = 1$ .

Hence  $(0, 6, 5) = (1, 1, 0) + 4(5, 2, 3)$ .

Let  $\mathbf{a} = (1, 1, 0)$  and  $\mathbf{b} = 4(5, 2, 3) = (6, 1, 5)$ .

Then  $A = \langle \mathbf{a} \rangle$ ,  $B = \langle \mathbf{b} \rangle$  and  $C = \langle \mathbf{a} + \mathbf{b} \rangle$ .

Let  $(4, 5, 6) = x(1, 1, 0) + y(6, 1, 5)$ .

$$\text{Then } \begin{cases} x + 6y = 4 \\ x + y = 5 \\ 5y = 6 \end{cases}.$$

The last equation gives  $5y = 6 = 13 = 20$ , so

$$y = 4 \text{ and } x = 1.$$

Thus  $(4, 5, 6) = (1, 1, 0) + 4(6, 1, 5) = \mathbf{a} + 4\mathbf{b}$ .

Dividing by 4, mod 7, is equivalent to multiplying by 2 since  $2 \cdot 4 = 1 \pmod{7}$ .

So  $2(4, 5, 6) = (1, 3, 5) = 2\mathbf{a} + \mathbf{b}$  and so  $D = \langle 2\mathbf{a} + \mathbf{b} \rangle$ .

Hence  $\mathfrak{R}(A, B; C, D) = 2$ .

#### Exercise 4:

Let  $A = \langle \mathbf{a} \rangle$ ,  $B = \langle \mathbf{b} \rangle$ ,  $C = \langle \mathbf{a} + \mathbf{b} \rangle$ ,  $D = \langle \lambda \mathbf{a} + \mathbf{b} \rangle$

where  $\lambda = \mathfrak{R}(A, B; C, D)$ .

Then  $D = \langle x\mathbf{a} + y\mathbf{b} \rangle$  for some scalars  $x, y$  where  $y \neq 0$  and

so  $D = \langle \mu \mathbf{a} + \mathbf{b} \rangle$  where  $\mu = x/y$ .

Then  $\mu = \mathfrak{R}(A, B; C, E)$ .

Since  $A, B, C, D, E$  are distinct  $\lambda, 1 - \lambda, \mu, 1 - \mu$  are all non-zero, so the fractions in the following all exist.

We want to find  $x, y$  such that  $\lambda \mathbf{a} + \mathbf{b} = x\mathbf{b} + y(\mathbf{a} + \mathbf{b})$ .

Then  $y = \lambda$  and  $x + y = 1$ , which gives:

$$y = \lambda \text{ and } x = 1 - \lambda.$$

So  $\lambda \mathbf{a} + \mathbf{b} = (1 - \lambda)\mathbf{b} + \lambda(\mathbf{a} + \mathbf{b})$ .

If  $\mathbf{b}' = (1 - \lambda)\mathbf{b}$  and  $\mathbf{c}' = \lambda(\mathbf{a} + \mathbf{b})$  then  $B = \langle \mathbf{b}' \rangle$ ,  $C = \langle \mathbf{c}' \rangle$  and  $D = \langle \mathbf{b}' + \mathbf{c}' \rangle$ .

We now want to write  $\mu \mathbf{a} + \mathbf{b}$  as a linear combination of  $\mathbf{b}'$  and  $\mathbf{c}'$ .

$$\text{Now } \mathbf{b} = \frac{1}{1 - \lambda} \mathbf{b}' \text{ and } \lambda \mathbf{a} = \mathbf{c}' - \lambda \mathbf{b} = \mathbf{c}' - \frac{\lambda}{1 - \lambda} \mathbf{b}'$$

$$\begin{aligned} \text{so } \mu \mathbf{a} + \mathbf{b} &= \left( \frac{\mu}{\lambda} \right) \left( \mathbf{c}' - \frac{\lambda}{1 - \lambda} \mathbf{b}' \right) + \frac{1}{1 - \lambda} \mathbf{b}' \\ &= \left( \frac{1 - \mu}{1 - \lambda} \right) \mathbf{b}' + \left( \frac{\mu}{\lambda} \right) \mathbf{c}'. \end{aligned}$$

Hence  $E = \left\langle \left( \frac{1 - \mu}{1 - \lambda} \right) \left( \frac{\lambda}{\mu} \right) \mathbf{b}' + \mathbf{c}' \right\rangle$  and so

$$\Re(B, C; D, E) = \left( \frac{1 - \mu}{1 - \lambda} \right) \left( \frac{\lambda}{\mu} \right) = \left( \frac{1 - \mu}{\mu} \right) \left( \frac{\lambda}{1 - \lambda} \right).$$

$$\text{Now } \left( \frac{1 - \mu}{\mu} \right) = - \left( 1 - \frac{1}{\mu} \right) \text{ and } \left( \frac{1 - \lambda}{\lambda} \right) = - \left( 1 - \frac{1}{\lambda} \right).$$

Finally  $\Re(A, B; C, D) = \lambda$  so  $\Re(A, B; D, C) = \frac{1}{\lambda}$  and

$$\text{hence } \Re(A, C; B, D) = 1 - \frac{1}{\lambda}.$$

Similarly  $\Re(A, B; C, E) = \mu$  so  $\Re(A, B; E, C) = \frac{1}{\mu}$  and

$$\text{hence } \Re(A, C; B, E) = 1 - \frac{1}{\mu}.$$

It follows that  $\mathfrak{R}(B, C; D, E) = \frac{\mathfrak{R}(A, C; B, E)}{\mathfrak{R}(A, C; B, D)}$ .

